# Frobenius Endomorphisms for Koszul Complexes


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## Notation

Throughout $R$ is a Noetherian ring with prime characteristic $p > 0$ and all modules/complexes are degreewise finitely-generated. Further all tensors are over $R$.

**Definition.** The Frobenius endomorphism $F\colon R \to R$ is the ring map $F(r) = r^p$.

Given an $R$-module $M$ the module obtained by restricting scalars along $F$, denoted $F_*M$, has elements of the form $F_*m$ for $m \in M$ and has the $R$-module structure

$$\begin{aligned} F_*m + F_*n &= F_*(m+n) && \text{for } m, n \in M \\ rF_*m &= F_*r^pm && \text{for } r \in R,\ m \in M \end{aligned}$$

If $A$ is an $R$-algebra then $F_*A$ is also an $R$-algebra with multiplication

$$F_*aF_*b = F_*ab \qquad \text{for } a, b \in A$$

We say $R$ is F-finite if the $R$-module $F_*R$ is finitely-generated.

The class of F-finite rings is closed under quotients, localizations, completions, and adjoining finitely-many polynomial or power series variables.

**Example.** For $k = \mathbb{Z}/p\mathbb{Z}$ and $R = k[x]$ we note $y = F_*x$ is a formal $p$th root of $x$ and so

$$F_*R \cong R[y]/(y^p - x) \cong R1 \oplus Ry \oplus \cdots \oplus Ry^{p-1}$$

is a finite rank free $R$-module.

For $R = k[x]/(x^2)$ we note $y$ is still a formal $p$th root with the additional relation $y^2 = 0$ showing

$$F_*R \cong R[y]/(y^p - x, y^2) \cong R[y]/(y^2, x) \cong k \oplus ky$$

is still finitely-generated but not free.

## Motivation

For F-finite rings there are many "Kunz-like" theorems characterizing commutative algebra properties using homological invariants:

**Kunz Like Theorem.** Let $R$ be an F-finite ring, $P$ a ring-theoretic property, and Hdim a homological invariant. Let $F^e$ denote the $e$-fold composition of $F$, i.e. $F^e(r) = r^{p^e}$. Then the following are equivalent:

i. $R$ is $P$,
ii. $\operatorname{Hdim}(F^e_*R) < \infty$ for some $e \gg 0$, and
iii. $\operatorname{Hdim}(M) < \infty$ for all finitely-generated $R$-modules.

Some examples include:

- (Kunz '69) $P$ = regular and Hdim = $\operatorname{pd}_R$ ($e = 1$).
- (Blanco-Majadas '98) $P$ = locally complete intersection and Hdim = $\text{CI-}\dim_R$ ($e = 1$).
- (Takahashi-Yoshino '04) $P$ = Gorenstein and Hdim = $\operatorname{Gpd}_R$.

These results can be uniformly explained using the derived category.

**Definition.** The category $\mathrm{D}^f(R)$ is obtained by taking the category of bounded chain complexes of finitely-generated $R$-modules and formally adding inverses to every quasi-isomorphism.

For two complexes $M, N \in \mathrm{D}^f(R)$, we say the level of $N$ against $M$ is $n$, denoted $\operatorname{level}^M_R(N) = n$, if $n$ is the smallest integer where $N$ admits a filtration in $\mathrm{D}^f(R)$ of the form

$$0 \subseteq X_1 \subseteq \cdots \subseteq X_n \simeq N \oplus Y$$

where each subquotient is a finite direct sum of shifts of $M$, i.e. with the form

$$X_{i+1}/X_i \cong \bigoplus_{j=1}^{c_i} \Sigma^{d_{ij}} M^{\oplus e_{ij}}$$

**Example.** For an $R$-module $M$, if $\operatorname{pd}_R(M) < \infty$ then any bounded and degreewise finitely-generated projective resolution $P$ defines a filtration by truncations

$$0 \subseteq P_{\leq 0} \subseteq \cdots \subseteq P_{\leq n} = P \simeq M$$

showing $\operatorname{level}^R_R(M) \leq \operatorname{pd}_R(M) + 1$.

For a module over a local ring $(R, \mathfrak{m}, k)$ with finite Loewy length then the $\mathfrak{m}$-adic filtration

$$0 = \mathfrak{m}^n M \subseteq \mathfrak{m}^{n-1} M \subseteq \cdots \subseteq \mathfrak{m} M \subseteq M$$

shows $\operatorname{level}^k_R(M) \leq \ell\ell_R(M)$.

The converse of each inequality also holds.

**Heuristic.** If $\operatorname{level}^M_R(N) < \infty$, then $\operatorname{Hdim}(M) < \infty$ implies $\operatorname{Hdim}(N) < \infty$.

**Definition.** $G \in \mathrm{D}^f(R)$ is a generator if $\operatorname{level}^G_R(M) < \infty$ for all $M \in \mathrm{D}^f(R)$.

Using the heuristic, if a generator $G$ has finite Hdim then every $M \in \mathrm{D}^f(R)$ must as well. So the "Kunz-like" theorems can be explained by showing $F^e_*R$ is a generator.

**Theorem** (Ballard-Iyengar-Lank-Mukhopadhyay-Pollitz '26)**.** Let $R$ be F-finite, then $F^e_*R$ is a generator for $\mathrm{D}^f(R)$ for $e \gg 0$. Moreover if $R$ is a locally complete intersection then $F_*R$ is a generator.

There are two key ingredients to the proof:

1. A local-to-global principal from Benson-Iyengar-Krause '15:

$$F^e_*R \text{ is a generator for } \mathrm{D}^f(R) \iff \operatorname{level}^{F^e_*R_{\mathfrak{p}}}_R(\kappa(\mathfrak{p})) < \infty$$

2a. In the general case simplicial methods are used to show the level is bounded above by $\operatorname{edim}(R_{\mathfrak{p}})+1$

2b. In the locally complete intersection case cohomological supports and a theorem of Stevenson '14 are used to show the level against $F_*R_{\mathfrak{p}}$ is finite without constructing an upper bound.

**Question.** Can $\operatorname{level}^{F^e_*(R)}_R(k)$ be explicitly bounded for an F-finite local complete intersection $(R, \mathfrak{m}, k)$? And is there an analogue for the simplicial methods using differential-graded algebras?

## Rings to DGAs

**Definition.** A differential graded algebra $R$-algebra, shortened to DGA, is a graded $R$-algebra $A^\natural$ together with a square-zero degree $-1$ $R$-linear map $\partial^A\colon A \to A$ satisfying the Leibniz rule

$$\partial^A(ab) = \partial^A(a)b + (-1)^{|a|}a\partial^A(B)$$

A (left) differential graded $A$-module, shortened to DG $A$-module, is a graded $A^\natural$-module $M^\natural$ together with a square-zero degree $-1$ $R$-linear map $\partial^M\colon M \to M$ satisfying the Leibniz rule

$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|}a\partial^M(m)$$

Every DG $A$-module is an $R$-complex and morphisms between DG $A$-modules are $R$-linear chain maps. $\mathrm{D}^f(A)$ is then the derived category of DG $A$-modules whose homology is finitely-generated over $H(A)$.

**Definition.** The Koszul complex over $R$ on a sequence of elements $f_1, \ldots, f_c \in R$ is the DGA $E = K^R(f_1, \ldots, f_c)$ given by

$$0 \longrightarrow \textstyle\bigwedge^c R^c \longrightarrow \bigwedge^{c-1} R^c \longrightarrow \ldots \longrightarrow \bigwedge^1 R^c \longrightarrow R \longrightarrow 0$$

whose multiplication is the wedge product and whose differential is

$$\partial(e_{i_1}\ldots e_{i_n}) = \sum_{j=1}^n (-1)^{j+1} f_{i_j} e_{i_1} \ldots e_{i_{j-1}} e_{i_{j+1}} \ldots e_{i_n}$$

We define the Frobenius endomorphism for $E$ to be the morphism of DG rings $F^E\colon E \to E$ given by

$$F^E(r) = r^p \text{ for } r \in R \qquad F^E(e_i) = f_i^{p-1}e_i$$

This endomorphism is recovered from translating the natural Frobenius endomorphism on the simplicial resolution of $E$ along the DGA homotopy equivalences.

Given a DG $E$-module $M$, the restriction of scalars along $F^E$ yields the DG $E$-module $F_*M$ whose component $R$-modules are $F_*M_n$ together with the differential and $E$-action

$$\begin{gathered} \partial\,(F_*x) = F_*\partial(x) \\ e \cdot F_*x = F_*F(e)x = F_*f^{p-1}ex \end{gathered}$$

**Example.** Let $k = \mathbb{Z}/3\mathbb{Z}$ and let $R = k[x]$. Let $f = x^2$, then $E = K^R(x^2)$ is the complex

$$0 \to Re \xrightarrow{x^2} R1 \to 0$$

The Frobenius endomorphism $F^E$ for $E$ extends $F^R$ by

$$F^E(e) = (x^2)^{3-1}e = x^4e$$

From the earlier example letting $y = F_*x$ we see $F_*R \cong R \oplus Ry \oplus Ry^2$ as an $R$-module. Letting $\xi = F_*e$ we see the differential and $E$-action are given by

$$\partial(y^i\xi) = y^{i+2} = \begin{cases} y^2, & i = 0 \\ xy^{i-1}, & i = 1, 2 \end{cases} \qquad e \cdot y^i = y^{4+i}\xi = \begin{cases} xy^{1+i}\xi, & i = 0, 1 \\ x^2\xi, & i = 2 \end{cases}$$

so all together $F_*E$ is the DG $E$-module

$$\begin{array}{c} R\xi \\ \oplus \\ Ry\xi \\ \oplus \\ Ry^2\xi \end{array} \ \underset{e = \begin{bmatrix} 0 & 0 & x^2 \\ x & 0 & 0 \\ 0 & x & 0 \end{bmatrix}}{\overset{\partial = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 1 & 0 & 0 \end{bmatrix}}{\rightleftarrows}} \ \begin{array}{c} R1 \\ \oplus \\ Ry \\ \oplus \\ Ry^2 \end{array}$$

Letting $k = R/(x)$ we see $\overline{1}$ and $\overline{y}$ define $k$-summands of $k \otimes F_*E$ as DG $E$-modules.

Moreover $F_*E$ is semifree as an $R$-complex meaning there is the quasi-isomorphism of DG $E$-modules

$$K^R(x) \otimes F_*E \xrightarrow{\simeq} k \otimes F_*E = k \oplus M$$

and thus a filtration

$$0 \subseteq F_*E \subseteq K^R(x) \otimes F_*E \simeq k \oplus M$$

showing $\operatorname{level}^{F_*E}_E(k) \leq 2$.

**Main Lemma** (Kalir '26)**.** Let $(R, \mathfrak{m}, k)$ be an F-finite regular local ring and let $f \neq 0 \in \mathfrak{m}$. Let $E = K^R(f)$ and $F$ the Frobenius endomorphism on $E$. Then $\operatorname{level}^{F_*E}_E(k) \leq \operatorname{edim}(R) + 1 < \infty$.

**Main Theorem** (Kalir '26)**.** Let $R$ be an F-finite locally hypersurface ring, then for every $\mathfrak{p} \in \operatorname{Spec}(R)$ we have $\operatorname{level}^{F_*R_{\mathfrak{p}}}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p})) \leq \operatorname{edim}(R) + 1 < \infty$.

Liu-Pollitz '25 develops a local-to-global principal for general Koszul complexes over a regular ring, so with a similar proof we also obtain:

**Main Corollary** (Kalir '26)**.** Let $R$ be an F-finite regular local ring, let $E = K^R(f_1, \ldots, f_c)$ be the Koszul complex on some sequence of nonzero elements $f_1, \ldots, f_c \in R$, and let $E_i = K^R(f_i)$. Then $F_*E_1 \otimes \cdots \otimes F_*E_c$ is a generator for $\mathrm{D}^f(E)$.

## References


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